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Existence of singular solutions to semilinear critical problems in R^2 (*dedicated to Prof. Adimurthi on the occasion of his 60th birthday*)

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We will mainly be concerned by the following semilinear critical problem :

$$(\mathrm{P}_\lambda)\begin{cases} -\Delta u = \lambda h(u) e^{u^2} \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega,\ u > 0 \text{ in } \Omega \end{cases}$$

i) Ω is a bounded domain in $\boldsymbol{R}^2$ with smooth boundary, $\lambda > 0$, $h \in C^1(\boldsymbol{R}^+, \boldsymbol{R}^+)$.

ii) *critical behaviour :* For any $\epsilon > 0$, $\lim_{t\to\infty} h(t)e^{-\epsilon t^2} = 0$, $\lim_{t\to+\infty} h(t)e^{\epsilon t^2} = +\infty$.

Depending on the perturbation h, we aim to discuss

- existence of weak bounded (and then classical) solutions,
- behaviour of solutions in respect to λ ($\to$ bifurcation diagram, concentration phenomena, existence of singular solutions),

Theorem

(Embedding in Orlicz space, Trudinger-Moser)

1. *Let $p < \infty$ then $u \in H_0^1(\Omega)$ implies $e^{u^2} \in L^p(\Omega)$ and is continuous in the norm topology.*
2. $$4\pi = \max\left\{ c\,;\ \sup_{\|w\|\leq 1} \int_\Omega e^{c|w|^2} < +\infty \right\}. \qquad (1)$$

Lack of compactness $\to$ $H_0^1(\Omega) \ni u \hookrightarrow e^{4\pi u^2} \in L^1(\Omega)$ is not compact for the weak topology. Considering the Moser functions ϕ_n

$$\phi_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{\frac{1}{2}} \text{ for } |x| \leq \frac{1}{n}, \\ \frac{\log(\frac{1}{r})}{(\log n)^{\frac{1}{2}}} \text{ for } \frac{1}{n} \leq |x| \leq 1, \\ 0 \text{ for } |x| \geq 1, \end{cases}$$

we have $\phi_n \rightharpoonup 0$ in $H_0^1(\Omega)$ as $n \to +\infty$.

Remarks.

1. P.L. Lions' result (concentration compactness principle) :

Theorem

Let $\{u_m\}$ be a sequence in $H_0^1(\Omega)$ such that $\|u_m\|_{H_0^1(\Omega)} \leq 1$ for all m. We may assume that $u_m \rightharpoonup u$ in $H_0^1(\Omega)$, $|\nabla u_m|^2 \to \mu$ weakly in measure. Then either : (i) $\mu = \delta_{x_0}$ and $u \equiv 0$, or (ii) there exists $\alpha > 4\pi$ such that $\{U_m \stackrel{\text{def}}{=} e^{u_m^2}\}$ is uniformly bounded in $L^\alpha(\Omega)$. (ii) holds if $u \not\equiv 0$.

2. Similar results for higher dimensions : Setting ω_N the volume of the sphere in $\boldsymbol{R}^N$,

$$N\omega_N^{\frac{1}{N-1}} = \max\left\{c\,;\ \sup_{\|w\|_{W_0^{1,N}(\Omega)}\leq 1} \int_\Omega e^{c|w|^{\frac{N}{N-1}}} < +\infty\right\}.$$

3. By Vitali's convergence theorem, compactness for $\alpha < \frac{N}{N-1}$.

Analogy with the critical exponent problem :

$$N \geq 3, \quad H_0^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$$

The scaling

$$u \to \alpha^{\frac{N-2}{2}} u(\alpha x) \stackrel{\text{def}}{=} u_\alpha(x)$$

makes invariant $\|\nabla u\|_{L^2(\Omega)}$ and $\int_\Omega u^{\frac{2N}{N-2}} \to$

$$S \stackrel{\text{def}}{=} \sup \left\{ \int_\Omega |u|^{2^*} \mathrm{d}x \backslash u \in H_0^1(\Omega) \, : \, \|u\|_{H_0^1(\Omega)} \leq 1 \right\}$$

is not achieved (for $N \geq 3$) and failure of the Palais Smale Condition for the functional E_λ

$$E_\lambda(u) \stackrel{\text{def}}{=} \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda u^2) \mathrm{d}x - \frac{1}{2^*} \int_\Omega |u|^{2^*} \mathrm{d}x$$

due to the lack of compactness (concentration phenomena).

Universal mechanism for (P.-S) sequences ("bubbling off spheres") :

Theorem

(Struwe) Let $\{u_m\}$ be a (P.-S.)-sequence for E_λ in $H_0^1(\Omega)$. Then there exist a number $k \in \mathbb{N}$, sequences (R_m^j), (x_m^j), $1 \le j \le k$, of radii $R_m^j \to \infty$ (as $m \to \infty$) and points $x_m^j \in \Omega$, a solution $u^0 \in H_0^1(\Omega)$ to the associated Euler Lagrange equation and non-trivial solutions $u^j \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $1 \le j \le k$, to the "limiting problem" $-\Delta u = u|u|^{2^*-2} \quad \text{in } \mathbb{R}^N$, such that a subsequence (u_m) satisfies $\|u_m - u^0 - \sum_{j=1}^{k} u_m^j\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \to 0$. Here u_m^j denotes the rescaled function

$$u_m^j(x) \stackrel{\text{def}}{=} (R_m^j)^{\frac{N-2}{2}} u^j(R_m^j(x - x_m^j)), \quad 1 \le j \le k, \; m \in \mathbb{N}.$$

Moreover, $E_\lambda(u_m) \to E_\lambda(u^0) + \sum_{j=1}^{k} E_0(u^j)$.

Remark.

if $u_m \geq 0$, then $E_0(u^j) = \frac{1}{N} S^{\frac{N}{2}}$ and (P.-S) condition hold at level β not in the form $\beta = E_\lambda(u^0) + \frac{k}{N} S^{\frac{N}{2}}$, $k \in \mathbb{N} \backslash \{0\}$. See works of Struwe, [Brezis, Nirenberg], [Bahri, Coron], [Rey].

Concerning the two dimension case,

1) [Adimurthi, Prashanth] $\rightarrow$ non universal behaviour of (P.-S.)-sequences : (P.-S.)- sequences exhibit different blow-up behaviour (Moser functions).

2) [Carleson, Chang], [Flücher] $\rightarrow$

$$\sup_{\|u\|_{H_0^1(\Omega)} \leq 1} \int_\Omega e^{4\pi u^2} \mathrm{d}x \text{ is achieved.}$$

Open problem raised by J. Moser : is the minimizer unique ?

Question : Existence of solutions to Problem (P_λ) regarding the asymptotic behaviour of the perturbation h.
Adimurthi $\rightarrow$
Let h satisfy in addition $h(0) = 0$, $h'(0) \in (0, \lambda_1(\Omega))$ and the asymptotic behaviour

$$\limsup_{t \to +\infty} h(t)t = \infty. \tag{2}$$

Then, There exists a non-trivial solution to (P_λ).
Remark. The solution is obtained by variational methods. (2) implies that there exists a (P.-S.)-sequence $\{u_m\}$ such that $\sup_m E_\lambda(u_m) <$ the first critical level where (P.-S.)-condition fails.
Sharpness of Condition (2) : the radial case ([De Figueiredo, Ruf], [Adimurthi, Prashanth])

Let $\Omega = B_1 \stackrel{\text{def}}{=} \{x \in \boldsymbol{R}^2 \backslash |x| < 1\}$, $\lambda = R^2$

$$(P_R) \qquad \left\{ \begin{array}{c} \left. \begin{array}{rcl} -\Delta u & = & f(u) \stackrel{\text{def}}{=} h(u)e^{u^2} \\ u & > & 0 \end{array} \right\} \text{ in } B_R, \\ u = 0 \;\text{ on } \partial B_R, u \in C^2(B_R). \end{array} \right.$$

From [Gidas, Ni, Nirenberg] result,it is sufficient to study $w(r) = u(|x| = r)$ for $r \in (0, R)$:

$$(P_R) \qquad \left\{ \begin{array}{c} \left. \begin{array}{rcl} -(rw')' & = & rf(w) \\ w & > & 0 \end{array} \right\} \text{ in } (0, R), \\ w'(0) = w(R) = 0. \end{array} \right.$$

Shooting method : $w(\cdot) = w(\cdot, \gamma)$: $w'(0, \gamma) = 0$, $w(0, \gamma) = \gamma$.
For $\gamma > 0$, denote $R_0(\gamma) \in (0, \infty)$ the first zero of $w(\cdot, \gamma)$.

Use the Emden-Fowler transformation and the [Atkinson-Peletier] Analysis :
$r = 2e^{-\frac{t}{2}}$, $y(t,\gamma) = w(r,\gamma)$ that yields

$$(P_\gamma) \qquad \begin{cases} -y'' = e^{-t} f(y), \\ y(\infty) = \gamma, y'(\infty) = 0. \end{cases}$$

Set $T_0(\gamma) \stackrel{\mathrm{def}}{=} 2\log\left(\frac{2}{R_0(\gamma)}\right)$.

Lemma

$T_0(\gamma)$ (hence $R_0(\gamma)$) is continuous function of γ in $(0,\infty)$ and strictly decreasing for small $\gamma > 0$. Futhermore, if $h(0) = 0$,
$$\lim_{\gamma \to 0^+} R_0(\gamma) = \frac{\lambda_1(B_1)}{h'(0)}.$$

Consequence : Existence/nonexistence of solutions to $(\mathrm{P}_\lambda) \leftrightarrow$ Asymptotics of $R_0(\gamma)$ as $\gamma \to \infty$!

Consider the following classes of perturbations h :

$$\mathcal{H}_1 = \left\{ h : \ h \text{ is asymptotic at } \infty \text{ to } e^{-t^{\beta}},\ 1 < \beta < 2 \right\},$$

$$\mathcal{H}_2 = \left\{ h : \ h \text{ is asymptotic at } \infty \text{ to } e^{-t^{\beta}},\ 0 < \beta \leq 1 \right\} \cup \{ h : \ h \text{ decays polynomially at } \infty \}$$

[Adimurthi, Prashanth] :

1. $h \in \mathcal{H}_1 \Rightarrow \limsup_{\gamma \to \infty} T_0(\gamma) < \infty \to$ nonexistence of solutions to (P_λ) for small λ.
2. $h \in \mathcal{H}_2 \Rightarrow \limsup_{\gamma \to \infty} T_0(\gamma) < \infty \to$ nonxistence of solutions to (P_λ) for small $\lambda > 0$. If $h'(0) = 0$ in addition, then existence of multiple solutions for large $\lambda > 0$.

Remark. [Adimurthi, Yadava] $\to$ condition $\limsup_{\gamma \to \infty} \dfrac{\log(h(t))}{t} = \infty$ is sharp to get existence of changing sign solutions.

Consider "sublinear" nonlinearities (i.e. either $h(0) > 0$ or $h(0) = 0$ and $h'(0) = \infty$). Then, one has existence of minimal solutions for small $\lambda > 0$ by lower and upper solutions technique (or $\lim_{\gamma \to 0^+} R_0(\gamma) = 0$).

Question : for which class of h, do we get multiplicity of solutions for all small λ (i.e. large solutions for small $\lambda > 0$) ?

From [Adimurthi-G] $\rightarrow$

(i) $h \in \mathcal{H}_1 \Rightarrow \liminf_{\gamma \to \infty} R_0(\gamma) > 0 \rightarrow$ uniqueness of solutions to (P_λ) for all small $\lambda > 0$;

(ii) $h \in \mathcal{H}_2 \Rightarrow \lim_{\gamma \to \infty} R_0(\gamma) = 0 \rightarrow$ multiplicity of solutions to (P_λ) for all small $\lambda > 0$

<u>Sharp condition ensuring multiplicity</u> : $\liminf_{t \to \infty} h(t) t e^{t^\beta} = \infty$ for some $\beta \in [0, 1]$.

To establish the above result, following the Atkinson-Peletier-ODE analysis, we need delicate estimates on asymptotics on y, y' at suitable points

- $T_1(\gamma) = \log(f(\gamma)) + \log\left(\frac{g'(\gamma)}{2}\right) - k\log(\gamma)$, $k \in \mathbb{N}$ large ;
- $T_2(\gamma)$ such that $y(T_2(\gamma)) = s_0$ where s_0 is large enough to get $\log f$ convex in $[s_0, \infty)$;
- $T_3(\gamma) = k' \log(\gamma)$ with $k' \in \mathbb{N}$ large enough.

Remark. Extension of [Adimurthi, G] to the N-laplace equation ($N > 2$) in [Prashanth, Sreenadh, G] with similar methods (variational methods to get multiplicity and ODE techniques to get sharpness of condition : $\liminf_{t\to\infty} h(t)te^{\epsilon t^{\frac{1}{N-1}}} = \infty$ for any $\epsilon > 0$ ensuring multiplicity).

Blow-up Pattern of sequences of solutions :

Contributions : [Struwe], [Ogawa, Susuki], [Adimurthi, Prashanth] (radial case), [Adimurthi, Struwe] (small energies), [Druet], [Adimurthi-Yang] (N-Laplace case).

Scaling argument : Let u_R be a (radial) solution to

$$\begin{cases} -\Delta u & = f(u) \stackrel{\text{def}}{=} h(u)e^{u^2} \quad \text{in } B_R, \\ u|_{\partial\Omega} & = 0, \ u(0) = \gamma > 0 \end{cases}$$

and for $\epsilon > 0$, set v defined for $r \in [0, R]$ by

$$v_(r) \stackrel{\text{def}}{=} g'(\gamma)(u_R(\epsilon r) - \gamma).$$

For a suitable ϵ, v satisfies $-\Delta v = e^v e^h$ in $B_{\frac{R}{\epsilon}}$ with $h = \frac{g''(\psi)v^2}{2g'(\gamma)}$.
$v \in L^\infty_{\text{loc}}$ and $v \to v_\infty$ in $C^2_{\text{loc}}(\boldsymbol{R}^2)$ where

$$-\Delta v_\infty = e^{v_\infty} \quad \text{in } \boldsymbol{R}^2, \ \int_{\boldsymbol{R}^2} e^{v_\infty} \ < \ \infty.$$

From [Chen, Li], $v_\infty = \log\left(\frac{32\lambda^2}{(4+\lambda^2|x-x_0|^2)^2}\right)$.

For nonlinearities satisfying (2)

Theorem

(Adimurthi, Prashanth)

(i) $\|u_R\|_{L^2(B_R)} \to 1$ *as* $R \to 0$,

(ii) *(blow up profile) There exists* $\rho = \rho(R)$, $\rho \to 0$ *as* $R \to 0$, *such that for any* $z \in \partial B_1$ *and* $x \in \mathbf{R}^2$ *:*

$$u_R^2(\rho x) - u_R^2(\frac{\rho z}{2}) \to 2\log\left(\frac{2}{1+|x|^2}\right)$$

as $\rho \to 0$ *uniformly on compact subsets of* $\mathbf{R}^2$.

Extensions for the non radial case and small energies in [Adimurthi, Struwe].

Druet $\to$ u_λ critical point to E_λ

$$E_\lambda(u_\lambda) = 4k\pi + o(1) \quad \text{as} \quad \lambda \to 0.$$

Back to sublinear nonlinearities (in the radial case, $\Omega = B_1$) :
$h(0 > 0$ or $h(0) = 0$ and $h'(0) = \infty$.
Consider first that h belongs to class $\mathcal{H}_2$.

Theorem

(Dhanya, G, Prashanth)

(i) *For every* $\{v_{\lambda_n}\}_{n\in\mathbb{N}}$ *a sequence of solutions respectively to* (P_{λ_n}) *with* $\lambda_n \to 0^+$ *and such that* $v_n(0) \to \infty$ *as* $n \to \infty$. *Then,* $v_n \to 0$ *as* $n \to \infty$ *locally uniformly in* $\overline{B}_1 \setminus \{0\}$ *and* $\|\nabla v_n\|_{L^2(B_1)} \to 4\pi$.

(ii) *Furthermore, there exists* $\rho_n \to 0^+$ *as* $n \to \infty$ *such that* $v_n^2(\rho_n r) - v_n^2(\rho_n) \to 2\log(2(1+r^2)^{-1})$ *in* $L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$.

Remarks.

1. Such $\{v_n\}$ exists (from Moutain Pass Theorem or ODE analysis : $\lim\limits_{\gamma\to\infty} R_0(\gamma) = 0$).
2. In this case, the limiting equation (as $n \to \infty$) is $-\Delta u = 2e^u$ in $\mathbb{R}^2$.

We now assume that h belongs to class $\mathcal{H}_1$. We recall that $\liminf_{\gamma\to\infty} R_0(\gamma) > 0$ as $\gamma \to \infty$.

Theorem

(Dhanya, Prashanth, G)

(i) *Let $\{v_{\lambda_n}\}$ be a sequence of solutions respectively to (P_{λ_n}) such that $\lambda_n \to \Lambda > 0$ and $v_n(0) \to \infty$ as $n \to \infty$. Then, $v_{\lambda_n} \to v^*$ uniformly on compact subsets of $B_1 \setminus \{0\}$ where v^* is a singular solution to (P_Λ).*

(ii) *$v^* \in L^q(\Omega)$ for any $1 \le q < \infty$. $v^* \notin L^\infty(\Omega)$ and $v^* \notin H_0^1(\Omega)$.*

Remark. From $\liminf\limits_{\gamma\to\infty} R_0(\gamma) > 0$, it can be proved that such sequence $\{v_n\}$ (and then singular solution v^*) exist.

Existence of the singular solution v^* :

Step 1 : *constructing* y^*. Let $\{\gamma_n\}$ such that $\gamma_n \to \infty$. Let $y_n \stackrel{\text{def}}{=} y(\cdot, \gamma_n)$. Set $T^* \stackrel{\text{def}}{=} \limsup T_0(\gamma_n) < \infty$. Then,

$\{y_n\}$ is a bounded sequence on compact subsets of $[T^*, \infty)$.

From Helly's Theorem and up to a subsequence (still denoted $\{\gamma_n\}$),

$$y^*(t) \stackrel{\text{def}}{=} \lim_{n\to\infty} y(t, \gamma_n) < \infty \quad \text{and } y^*(t) \to \infty \text{ as } t \to \infty.$$

Step 2 : y^* *solves the Emden-Fowler equation*. From the integral representation, $-(y^*)'' = f(y^*)$ in (T^*, ∞) with $\int_{T^*+2}^{\infty} f(y^*(t))e^{-t}\mathrm{d}t < \infty$.

Step 3 : *Back to the original problem*. Defining $R^* = 2e^{-\frac{T^*}{2}}$, going back to our original variable $x \in B_R$ and defining $v^*(x) = y^*(2\log(\frac{2}{|x|}))$ we obtain that v^* solves the following problem (P_R^*) :

$$\left\{ \begin{array}{c} \left. \begin{array}{rcl} -\Delta u &=& f(u) \\ u &>& 0 \end{array} \right\} \text{ in } B_{R^*} \setminus \{0\}, \\ \lim_{|x|\to 0} u(x) = \infty, \end{array} \right.$$

with $\int_{B_{R^*}} f(v^*) < \infty$.

Step 4 : *removable singularity*. [Brezis, Lions] $\rightarrow$ v^* solves the problem (P'_α) :

$$-\Delta u = f(u) + \alpha\delta_0$$

in the sense of distributions in B_{R^*} for some $\alpha \geq 0$.

Definition

We call f a sub-exponential type function if

$$\lim_{t\to\infty} f(t)e^{-\beta t} \leq C \quad \text{for} \quad \text{some} \quad \beta, \mathrm{C} > 0.$$

We call f to be of super-exponential type if it is not a sub-exponential type function.

Theorem

(Dhanya, G, Prashanth) *Let f be of super-exponential type. Then any solution u of (P'_α) extends to a distributional solution of (P_R).*

. Define $\beta^* = \inf\{\beta > 0 \ \text{occuring} \ \text{in} \ \text{Definition}\}$. Then,

Theorem

(Dhanya, G, Prashanth) *Let f be a sub-exponential type non-linearity. Then for all $\alpha \in (0, \frac{2\pi}{\beta^*})$ the problem (P'_α) admits a solution. Furthermore, if $f(t) \geq Ce^{\overline{\beta}t}, \forall t \geq 0$, for some $\overline{\beta} > 0$, then (P'_α) has no solution for all $\alpha \geq 4\pi(\overline{\beta})^{-1}$.*

End of the proof of assertion (ii) :
From above Theorems, $\alpha = 0$ and v^* is an unbounded weak solution to (P^*_R).
Therefore, from Trudinger-Moser imbedding, $v^* \notin H^1_0(\Omega)$.
From $\int_{B_R} f(v^*) < \infty$, $v^* \in L^q(\Omega)$ for any $1 \leq q < \infty$. □
Other properties of v^* :

1. By oscillation criterion, v^* has infinitely many zeroes in B^*_R accumulating at 0.
2. v^* has infinite Morse index.
3. In the analytic case, infinitetely many turning points ; contrast with perturbations of class $\mathcal{H}_2$ case.